Fast Estimator of Primordial Non-Gaussianity from Temperature and Polarization Anisotropies in the Cosmic Microwave Background

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ABSTRACT

Measurements of primordial non-Gaussianity (f_{NL}) open a new window onto the physics of inflation. We describe a fast cubic (bispectrum) estimator of f_{NL}, using a combined analysis of temperature and polarization observations. The speed of our estimator allows us to use a sufficient number of Monte Carlo simulations to characterize its statistical properties in the presence of real world issues such as instrumental effects, partial sky coverage, and foreground contamination. We find that our estimator is optimal, where optimality is defined by saturation of the Cramer Rao bound, if noise is homogeneous. Our estimator is also computationally efficient, scaling as $O(N^{3/2})$ compared to the $O(N^{5/2})$ scaling of the brute force bispectrum calculation for sky maps with N pixels. For Planck this translates into a speed-up by factors of millions, reducing the required computing time from thousands of years to just hours and thus making f_{NL} estimation feasible for future surveys. Our estimator in its current form is optimal if noise is homogeneous. In future work our fast polarized bispectrum estimator should be extended to deal with inhomogeneous noise in an analogous way to how the existing fast temperature estimator was generalized.

Subject headings: cosmic microwave background, early universe, inflation

1. Introduction

In the last few decades the advances in the observational cosmology have led the field to its "golden age." Cosmologists are beginning to nail down the basic cosmological parameters, and have started asking questions about the nature of the initial conditions provided by inflation (Guth 1981; Sato 1981; Linde 1982; Albrecht & Steinhardt 1982), which apart from solving the flatness and horizon problem, also gives a mechanism for producing the seed perturbations for structure

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formation (Guth & Pi 1982; Starobinsky 1982; Hawking 1982; Bardeen et al. 1983; Mukhanov et al. 1992), and other testable predictions.

The main predictions of a canonical inflation model are: (i) spatial flatness of the observable universe, (ii) homogeneity and isotropy on large scales of the observable universe, (iii) nearly scale invariant and adiabatic primordial density perturbations, and (iv) primordial perturbations to be very close to Gaussian. The cosmic microwave background (CMB) data from the Wilkinson Anisotropy Probe (WMAP) (Bennett et al. 2003), both temperature (Hinshaw et al. 2003, 2006) and polarization (Kogut et al. 2003; Page et al. 2006) anisotropies, have provided hitherto the strongest constraints on these predictions (Komatsu et al. 2003; Peiris et al. 2003; Spergel et al. 2003, 2006). There is no observational evidence against simple inflation models.

Non-Gaussianity from the simplest inflation models that are based on a slowly rolling scalar field is very small (Salopek & Bond 1990, 1991; Falk et al. 1993; Gangui et al. 1994; Acquaviva et al. 2003; Maldacena 2003); however, a very large class of more general models with, e.g., multiple scalar fields, features in inflaton potential, non-adiabatic fluctuations, non-canonical kinetic terms, deviations from Bunch-Davies vacuum, among others (Bartolo et al. 2004, for a review and references therein) generates substantially higher amounts of non-Gaussianity.

The amplitude of non-Gaussianity constrained from the data is often quoted in terms of a non-linearity parameter f_{NL} (defined in section 2.1). Many efficent methods for evealuating bispectum of CMB temperature anisotropies exist (Komatsu et al. 2005; Cabella et al. 2006a; Smith & Zaldarriaga 2006). So far, the bispectrum tests of non-Gaussianity have not detected any significant f_{NL} in temperature fluctuations mapped by COBE (Komatsu et al. 2002) and WMAP (Komatsu et al. 2003; Spergel et al. 2006; Creminelli et al. 2006a,b; Cabella et al. 2006b; Chen & Szapudi 2006). Different models of inflation predict different amounts of f_{NL}, starting from $O(1)$ to $f_{NL} \sim 100$, above which values have been excluded by the WMAP data already. On the other hand, some authors have claimed non-Gaussian signatures in the WMAP temperature data (Mukherjee & Wang 2004; Larson & Wandelt 2004; Vielva et al. 2004; Chiang et al. 2003, 2004). These signatures cannot be characterized by f_{NL} and are consistent with non-detection of f_{NL}.

Currently the constraints on the f_{NL} come from temperature anisotropy data alone. By also having the polarization information in the cosmic microwave background, one can improve sensivity to primordial fluctuations (Babich & Zaldarriaga 2004; Yadav & Wandelt 2005). Although the experiments have alrady started characterizing polarization anisotropies (Kovac et al. 2002; Kogut et al. 2003; Page et al. 2006; Montroy et al. 2006), the errors are large in comparison to temperature anisotropy. The upcoming experiments such as Planck will characterize polarization anisotropy to high accuracy. Are we ready to use future polarization data for testing Gaussianity of primordial fluctuations? Do we have a fast estimator which allows us to measure f_{NL} from the combined analysis of temperature and polarization data?

In this paper we extend the fast cubic estimator of f_{NL} from the temperature data (Komatsu et al.

2005) and derive a fast way for measuring primordial non-Gaussianity using the cosmic microwave background temperature and polarization maps. We construct a cubic statistics, a cubic combination of (appropriately filtered) temperature and polarization maps, which is specifically sensitive to the primordial perturbations. This is done by reconstructing a map of primordial perturbations, and using that to define our estimator. We also show that the inverse of the covariance matrix for the optimal estimator (Babich & Zaldarriaga 2004) is the same as the product of inverses we get in the fast estimator. Our estimator takes only $N^{3/2}$ operations in comparison to the full bispectrum calculation which takes $N^{5/2}$ operations. Here N refers to the total number of pixels. For Planck $N \sim 5 \times 10^7$, and so the full bispectrum analysis is not feasible while ours is.

2. Primordial Non-Gaussianity

2.1. A Model

The harmonic coefficients of the CMB anisotropy $a_{lm} = T^{-1} \int d^2\hat{\mathbf{n}} \Delta T(\hat{\mathbf{n}}) Y^{\star}_{\ell m}$ can be related to the primordial fluctuation as:

$$a^X_{\ell m} = \frac{2b_\ell}{\pi} \int k^2 dk\, r^2 dr\, [\, \Phi_{\ell m}(r)\, g^{adi}_{X\ell}(k) + S_{\ell m}(r)\, g^{iso}_{X\ell}(k)\,]\, j_\ell(kr) + n_{\ell m} \tag{1}$$

where $\Phi_{\ell m}(r)$ and $S_{\ell m}(r)$ are the harmonic coefficients of the primordial curvature perturbations and the primordial isocurvature perturbations respectively at a given comoving distance $r = |\mathbf{r}|$; $g_{X\ell}(r)$ is the radiation transfer function of either adiabatic or isocurvature perturbations; X refers to either T or E; and $j_\ell(kr)$ is the Bessel function of order ℓ. A beam function b_ℓ and the harmonic coefficient of noise $n_{\ell m}$ are instrumental and observational effects. Eq. (1) is written for flat background, but can easily be generalized.

Any non-Gaussianity present in the primordial perturbations $\Phi_{\ell m}(r)$ or $S_{\ell m}(r)$, can get transfered to the observed CMB i.e. a_{lm}, via Eq. (1). Due to the smallness of isocurvature contribution over the curvature perturbation (Peiris et al. 2003; Bean et al. 2006; Trotta 2006) we will drop the isocurvature contribution from Eq. (1) and further we will use a popular and simple non-Gaussianity model (Gangui et al. 1994; Verde et al. 2000; Komatsu & Spergel 2001) given by

$$\Phi(r) = \Phi_L(r) + f_{NL}[\Phi^2_L(r) - \langle \Phi^2_L(r) \rangle] \tag{2}$$

where $\Phi_L(r)$ is the linear Gaussian part of the perturbations, and f_{NL} is a non-linear coupling constatnt characterizing the amplitude of the non-Gausianity. The bispectrum in this model can be written as:

$$\langle \Phi(k_1)\Phi(k_2)\Phi(k_3) \rangle = 2f_{NL}(2\pi)^3 \delta^3(k_1 + k_2 + k_3)[P(k_1)P(k_2) + cycl.]. \tag{3}$$

The above form of the bispectrum is specific to the model chosen and so in general the constrains on f_{NL} do not necessarily tell us about non-Gaussianity of other models (Babich et al.

2004; Bartolo et al. 2004). This model, for instance, fails completely when non-Gaussianity is localized in a specific range in k space, the case that is predicted from inflation models with features in inflaton potential (Wang & Kamionkowski 2000; Komatsu et al. 2003; Chen et al. 2006). Even for the simplest inflation models based on a slowly rolling scalar field, the bispectrum of *inflaton perturbations* yields a non-trivial scale dependence of f_{NL} (Falk et al. 1993; Maldacena 2003), although the amplitude is too small to detect. On the other hand the bispectrum of *curvature perturbations* contains the contribution after the horizon exit, which is non-zero even when inflaton perturbations are exactly Gaussian. This contribution actually follows Eq. (3) (Lyth & Rodríguez 2005). Curvaton models also yield the bispectrum in the form given by Eq. (3) (Lyth et al. 2003).

2.2. Reconstructing Primordial Perturbations using Temperature and Polarization Anisotropy

Reconstruction of primordial perturbations from the cosmological data allows us to be more sensitive to primordial non-Gaussianity, which is important because non-Gaussianity in the cosmic microwave background data does not necessarily imply the presence of primordial non-Gaussianity. The method of reconstruction from temperature data is described in (Komatsu et al. 2005) and that from the combined analysis of temperature and polarization data is described in (Yadav & Wandelt 2005), where we reconstruct the perturbations $\Phi(r)$ using an operator $O_\ell(r)$. These operators are given by:

$$\begin{pmatrix} O_\ell^T(r) \\ O_\ell^E(r) \end{pmatrix} = \begin{pmatrix} C_\ell^{TT} & C_\ell^{TE} \\ C_l^{TE} & C_l^{EE} \end{pmatrix}^{-1} \begin{pmatrix} \beta_\ell^T(r) \\ \beta_\ell^E(r) \end{pmatrix}, \tag{4}$$

where

$$C_\ell^{XY} \equiv \langle a_{\ell m}^X a_{\ell m}^{*Y} \rangle = \frac{2b_\ell^X b_\ell^Y}{\pi} \int k^2 dk\, P_\Phi(k)\, g_{X\ell}\, g_{Y\ell}(k) + N_\ell^{XY} \tag{5}$$

$$\beta_\ell^X(r) \equiv \langle \Phi_{\ell m}(r) a_{\ell m}^{*X} \rangle = \frac{2b_\ell^X}{\pi} \int k^2 dk\, P_\Phi(k)\, g_{X\ell}(k)\, j_\ell(kr), \tag{6}$$

$P_\Phi(k)$ is the power spectrum of the primordial curvature perturbations, and $N_\ell^{XY} = \langle a_{\ell m}^X a_{\ell m}^{\star Y} \rangle$ is the noise covariance matrix. The primordial perturbation then can be estimated as:

$$\hat{\Phi}_{\ell m}(r) = \sum_{X=T,E} O_\ell^X(r)\, a_{\ell m}^X \tag{7}$$

Figure 1 shows the improvement in reconstruction due to additional information from the CMB polarization.

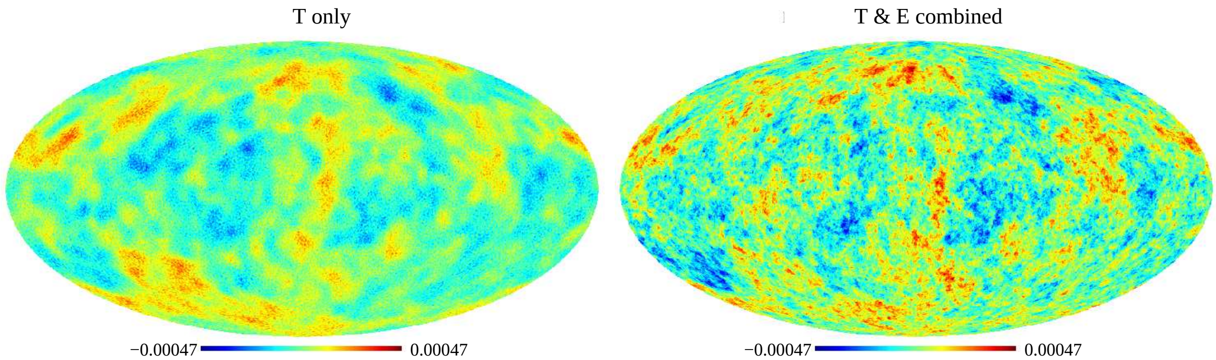

Fig. 1.— Reconstructed primordial perturbation map using only temperature information (left), and, using both temperature and polarization information (right). Perturbations are reconstructed at the surface of last scattering.

2.3. Fast Cubic Statistics

We can construct a quantity $\hat{S}_{prim}$ that is analogous to the KSW fast estimator of f_{NL} from temperature data (Komatsu et al. 2005) but generalize it to include polarization data as well. This quantity has a simple interpretation in terms of the tomographic reconstruction of the primoridal potential described in (Komatsu et al. 2005; Yadav & Wandelt 2005). It is the radial integral of a cubic combination of the scalar potential reconstructions, the B terms with the analogous A term.

As in (Komatsu et al. 2005), we form a cubic statistics given by

$$\hat{S}_{prim} = \frac{1}{f_{sky}} \int r^2 dr \int d^2\hat{n} B(\hat{n}, r) B(\hat{n}, r) A(\hat{n}, r) \tag{8}$$

where

$$B(\hat{n}, r) \equiv \sum_{ip} \sum_{lm} (C^{-1})^{ip} a^i_{\ell m} \beta^p_\ell(r) Y_{\ell m}(\hat{n}), \tag{9}$$

$$A(\hat{n}, r) \equiv \sum_{ip} \sum_{lm} (C^{-1})^{ip} a^i_{\ell m} \alpha^p_\ell(r) Y_{\ell m}(\hat{n}), \tag{10}$$

$\alpha^p_\ell = \frac{2b^p_\ell}{\pi} \int k^2 dk g_{X\ell}(k)\, j_\ell(kr)$, and f_{sky} is a fraction of sky. Index i and p can either be T or E. We find (see Appendix A) that $\langle \hat{S}_{prim} \rangle$ reduces to

$$\langle \hat{S}_{prim} \rangle = \frac{1}{f_{sky}} \sum_{ijkpqr} \sum_{2 \le \ell_1 \le \ell_2 \le \ell_3} \frac{1}{\Delta_{\ell_1 \ell_2 \ell_3}} f_{NL} B^{pqr,prim}_{\ell_1 \ell_2 \ell_3} (C^{-1})^{ip}_{\ell_1} (C^{-1})^{jq}_{\ell_2} (C^{-1})^{kr}_{\ell_3} B^{ijk,prim}_{\ell_1 \ell_2 \ell_3}, \tag{11}$$

where $\Delta_{\ell_1 \ell_2 \ell_3}$ is: 1 when $\ell_1 \neq \ell_2 \neq \ell_3$, 6 when $\ell_1 = \ell_2 = \ell_3$, and 2 otherwise; $B^{pqr,prim}_{\ell_1 \ell_2 \ell_3}$ is the theoretical bispectrum for $f_{NL} = 1$, and is given by:

$$B^{pqr,prim}_{\ell_1\ell_2\ell_3} = I_{\ell_1\ell_2\ell_3}\int r^2dr[\beta^p_{\ell_1}(r)\beta^q_{\ell_2}(r)\alpha^r_{\ell_3}(r) + \beta^r_{\ell_3}(r)\beta^p_{\ell_1}(r)\alpha^q_{\ell_2}(r) + \beta^q_{\ell_2}(r)\beta^r_{\ell_3}(r)\alpha^p_{\ell_1}(r)] \tag{12}$$

where

$$I_{\ell_1\ell_2\ell_3} = 2\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Since $\langle \hat{S}_{prim}\rangle$ is proportional to f_{NL}, an unbiased estimator of f_{NL} can be written as:

$$\hat{f}_{NL} = \frac{\hat{S}_{prim}}{\left\{\sum_{ijkpqr}\sum_{2\le\ell_1\le\ell_2\le\ell_3}\frac{1}{\Delta_{\ell_1\ell_2\ell_3}}B^{pqr,prim}_{\ell_1\ell_2\ell_3}(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}B^{ijk,prim}_{\ell_1\ell_2\ell_3}\right\}} \tag{14}$$

The most time consuming part of the calculation is the harmonic transformation necessary for Eqs. (9) and(10). The fast estimator defined above takes only $N^{3/2}$ operations times the number of sampling points for r, which is of the order 100. Hence this is much faster than the full bispectrum analysis, which scales as $N^{5/2}$. N here is the number of pixels. In the next section we will show that this fast unbiased estimator is also optimal, by proving equivalence with a known optimal but slow estimator.

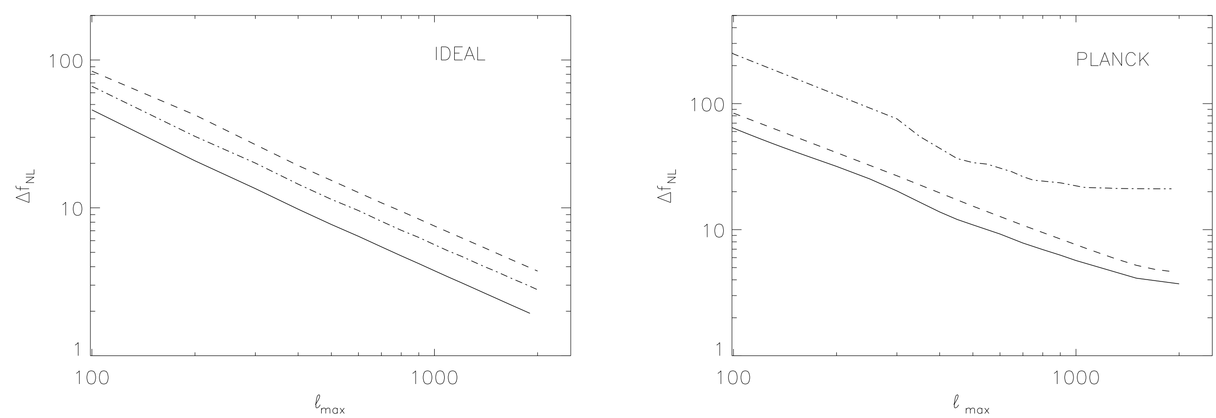

Fig. 2.— Fisher predictions for minimum detectable f_{NL} at the 1-σ level. Left panel: ideal experiment. Right panel: Planck satellite. Solid lines: temperature and polarization information combined. Dashed lines: temperature information only. Dot-dashed line: polarization information only.

2.4. Optimality of the fast estimator

In their recent paper Babich and Zaldarriaga (Babich & Zaldarriaga 2004) have found an optimal estimator for f_{NL} which minimizes the expected χ^2 given by

$$\chi^2 = \sum_{ijkpqr} \sum_{\ell_1\ell_2\ell_3} \left(B^{ijk,obs}_{\ell_1\ell_2\ell_3} - f_{NL} B^{ijk,prim}_{\ell_1\ell_2\ell_3} \right) (\mathbf{Cov}^{-1})^{ijk}_{pqr} \left(B^{pqr,obs}_{\ell_1\ell_2\ell_3} - f_{NL} B^{pqr,prim}_{\ell_1\ell_2\ell_3} \right). \tag{15}$$

This optimal estimator is

$$\hat{f}_{NL} = \frac{\sum_{ijkpqr} \sum_{\ell_1\ell_2\ell_3} B^{ijk,obs}_{\ell_1\ell_2\ell_3} (\mathbf{Cov}{-}1)_{ijk,pqr} B^{pqr,prim}_{\ell_1\ell_2\ell_3}}{\sum_{ijkpqr} \sum_{\ell_1\ell_2\ell_3} B^{ijk,prim}_{\ell_1\ell_2\ell_3} (\mathbf{Cov}^{-1})_{ijk,pqr} B^{pqr,prim}_{\ell_1\ell_2\ell_3}}, \tag{16}$$

where the index ijk and pqr runs over $\{TTT, TTE, TEE, EEE\}$, unlike in the fast estimator case where ijk and pqr run over all the 8 combinations $\{TTT, TTE, TET, ETT, TEE, ETE, EET, EEE\}$. In appendix B we show that the inverse of the covariance matrix for the BZ estimator is the same as the product of inverses we get in the fast estimator, Eq. 14; hence our estimator is optimal[1]

3. Results

To test optimality of our fast estimator we use Eqs. (14) and (8) to measure f_{NL} from simulated Gaussian skies. The error bars on f_{NL} are then derived from Monte Carlo simulations of our fast estimator. The simulated errors are compared with the Cramer Rao bound $(F^{-1}/f_{sky})^{1/2}$, where f_{sky} is a fraction of sky, and F is the Fisher matrix given by (Komatsu & Spergel 2001; Babich & Zaldarriaga 2004):

$$F = \sum_{ijkpqr} \sum_{\ell_1 \le \ell_2 \le \ell_3} \frac{1}{\Delta_{\ell_1\ell_2\ell_3}} B^{pqr,prim}_{\ell_1\ell_2\ell_3} (C^{-1})^{ip}_{\ell_1} (C^{-1})^{jq}_{\ell_2} (C^{-1})^{kr}_{\ell_3} B^{ijk,prim}_{\ell_1\ell_2\ell_3}, \tag{17}$$

where $\Delta_{\ell_1\ell_2\ell_3}$ is: 1 when $\ell_1 \neq \ell_2 \neq \ell_3$, 6 when $\ell_1 = \ell_2 = \ell_3$, and 2 otherwise. We have used ΛCDM model with $\Omega_c = 0.26, \Omega_b = 0.04, \Omega_\Lambda = 0.7, h = 0.7$, and a constant scalar spectral index $n_s = 1$. Since the contribution to the integral in Eqs. 9 and 10 comes mostly from the decoupling epoch (for our simulations $r_{dec} = 13.61$Gpc), our integration limits are $r_{min} = 13425$ Mpc and $r_{max} = 13865$ Mpc, with the sampling $dr = 15$ Mpc. Refining the sampling of the r integral by a factor of 2 does not change our results significantly. The results are summarized in figure 3. We separately explore the effects of excluding foreground contaminated sky regions and noise degradation due to the instrument. For definiteness we have used the published Planck noise amplitudes which are described in Table 2, though we ignore the effect of the scanning strategy and noise correlations. We shall come back to this point in § 4.

[1]By an analogous argument one can show that the temperature-only estimator in Komatsu et al. (2005) is also optimal, not slightly suboptimal as stated.

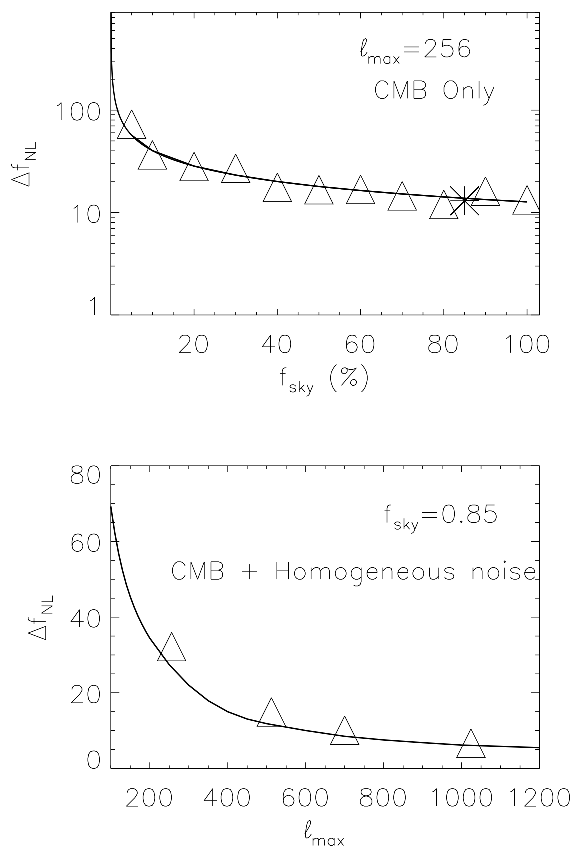

Fig. 3.— Testing optimality of our f_{NL} estimator. In all panels lines show the optimal Cramer Rao bounds given by $\left(F^{-1}/f_{sky}\right)^{1/2}$. Symbols show the 1-$\sigma$ errors on f_{NL} derived from Monte Carlo simulations. **Upper panels:** Monte Carlo errors as a function of f_{sky}. The star shows the WMAP Kp2 mask. Triangles: straight sky cuts excluding regions of low galactic latitudes. The left panel shows only the effect of the mask, while the right panel includes homogeneous white noise and beam smoothing at the level of the Planck satellite. **Lower left:** Monte Carlo errors as a function of ℓ_{max}. **Lower right:** Incomplete sky coverage causes excess variancde of the low ℓ modes of the polarization bispectra. We show results as a function of ℓ_{min}, below which multipoles have been removed from the analysis of polarization bispectra in two cases: 1) stars show the simluated errors in the noiseless case; 2) the dashed line and triangles show a case with homogeneous noise similar to the Planck satellite. The variance excess due to overweighting of the polarization modes is cleary visible in the noiseless case. This panel demonstrates that excluding the lowest ℓ polarization modes from the analysis avoids this variance excess without significant loss of information.

We find that the low ℓ modes of the polarization bispectrum are contaminated significantly when $f_{sky} < 1$. This is illustrated by the lower right panel of Figure 3, where we sum over all the ℓ modes for TTT contribution to the bispectrum (because the temperature bispectrum is not contaminated by the sky cut), while varying ℓ_{min} for the other terms (EEE, TTE, TEE). Our results show that one may simply remove the contamination in the polarization bispectrum due to the sky cut by removing $\ell \lesssim 10$ (when $f_{sky} \sim 0.8$) without sacrificing sensitivity to f_{NL}. The contamination appears to be less when noise is added, as the dominant constraint still comes from the temperature data which are insensitive to the contamination due to the sky cut.

We can explain this behaviour simply in terms of the coupling between spherical harmonic modes induced by the sky cut. The key observation is that the low ℓ polarization spectrum is very small in the theoretical model we chose, but if it could be observed it would add information to the f_{NL} estimator. Therefore, for a noiseless experiment, an optimal estimator designed for full sky work will assign a large weight to the low ℓ polarization modes. Since the low ℓ spectrum rises very steeply towards higher ℓ a sky cut creates power leakage from higher ℓ to lower ℓ. This biases the low ℓ spectrum significantly and this bias is amplified by the large coefficient the estimator assigns to these modes. In a realistic experiment including noise, the low ℓ polarization modes are difficult to measure, since noise dominates. Accordingly, the estimator assigns small weights to the low ℓ polarization modes and the sky cut has a much less prominent effect.

3.1. Computational Speed

Our fast estimator takes only $N^{3/2}$ operations times the number of sampling points for the integral over r, which is of the order 100. Hence this is much faster than the full bispectrum analysis as discussed in (Babich & Zaldarriaga 2004), which goes as $N^{5/2}$. N here is the number of pixels. For Planck we expect $N \sim 5 \times 10^7$, so performing 100 simulations using 50 CPUs takes only 10 hours using our fast estimator, while we estimate it would take approximately 10^3 years to do the brute force bispectrum calculation using the same platform.

Table 1: Planck noise properties assumed for our analysis

	Central frequency (GHZ)						
	30	44	70	100	143	217	353
Angular Resolution [FWHM arcminutes]	33	24	14	10	7.1	5.0	5.0
$\Delta T/T$ intensity [a] $[10^{-6}\mu$K/K$]$	2.0	2.7	4.7	2.5	2.2	4.8	14.7
$\Delta T/T$ polarization (Q and U)[a] $[10^{-6}\mu$K/K$]$	2.8	3.9	6.7	4.0	4.2	9.8	29.8

[a] Average 1σ sensitivity per pixel (a square whose side is the FWHM extent of the beam), in thermodynamic temperature units, achievable after 2 full sky surveys (14 months).

4. Conclusions

Starting with the tomographic reconstruction approach (Komatsu et al. 2005; Yadav & Wandelt 2005) we have found a fast, feasible, and optimal estimator of f_{NL}, a parameter characterizing the amplitude of primordial non-Gaussianity, based on three-point correlations in the temperature and polarization anisotropies of the cosmic microwave background. Using the example of the Planck mission our estimator is faster by factors of order 10^6 than the estimator described by Babich and Zaldarriaga (Babich & Zaldarriaga 2004), and yet provides essentially identical error bars.

The speed of our estimator allows us to study its statistical properties using Monte Carlo simulations. We have explored the effects of instrument noise (assuming homogeneous noise), finite resolution, as well as sky cut. We conclude that our fast estimator is robust to these effects and extracts information optimally when compared to the Cramer Rao bound, in the limit of homogeneous noise.

We have uncovered a potential systematic effect that is important for instruments measuring polarization with extremely high signal-to-noise on large scales. The inevitable removal of contaminated portions of the sky causes any estimator based on the pseudo-bispectrum to be contaminated by mode-to-mode couplings at low ℓ. We have demonstrated that by simply excluding low ℓ polarization modes from the analysis removes this systematic error with negligible information loss.

It has been shown that inhomogeneous noise causes cubic estimators based on the psudo-bispectrum with a flat weighting to be significantly suboptimal (Komatsu et al. 2003). A partial solution to this problem has been found by Creminelli et al. (2006a,b), where a linear piece has been added to the estimator in addition to the cubic piece. It should be straightforward to apply their method to our estimator.

Finally, our reconstruction approach may be extended to find fast estimators for higher order statistics, for example trispectrum based estimators of f_{NL} (Kogo & Komatsu 2006). This is the subject of ongoing work.

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5. Derivation of Fast Cubic Estimator

We derive an expectation value of the cubic statistics given by Eq. 8

$$\langle\hat{S}_{prim}\rangle = \frac{1}{f_{sky}}\int r^2 dr \int d^2\hat{n}\langle B(\hat{n},r)B(\hat{n},r)A(\hat{n},r)\rangle \tag{18}$$

using the form of B and A as given by Eq. 9 and 10

$$\langle\hat{S}_{prim}\rangle = \frac{1}{f_{sky}}\sum_{ijkpqr}\sum_{\ell_1\ell_2\ell_3}\sum_{m_1m_2m_3}(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}\langle a^i_{\ell_1m_1}a^j_{\ell_2m_2}a^k_{\ell_3m_3}\rangle\int r^2 dr\beta^p_{\ell_1}(r)\beta^q_{\ell_2}(r)\alpha^r_{\ell_3}(r)$$
$$\int d^2\hat{n}Y_{\ell_1m_1}(\hat{n})Y_{\ell_2m_2}(\hat{n})Y_{\ell_3m_3}(\hat{n}) \tag{19}$$

which simplifies to

$$\langle\hat{S}_{prim}\rangle = \frac{1}{f_{sky}}\sum_{ijkpqr}\sum_{\ell_1\ell_2\ell_3}I_{\ell_1\ell_2\ell_3}(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}\int r^2 dr\beta^p_{\ell_1}(r)\beta^q_{\ell_2}(r)\alpha^r_{\ell_3}(r)B^{ijk}_{\ell_1\ell_2\ell_3} \tag{20}$$

where

$$I_{\ell_1\ell_2\ell_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}, \tag{21}$$

and

$$B^{ijk}_{\ell_1\ell_2\ell_3} = \sum_{m_1m_2m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}B^{ijk}_{\ell_1\ell_2\ell_3m_1m_2m_3} \tag{22}$$

is the angular bispectrum, and $B^{ijk}_{\ell_1\ell_2\ell_3m_1m_2m_3} = \langle a^i_{\ell_1m_1}a^j_{\ell_2m_2}a^k_{\ell_3m_3}\rangle$is the CMB bispectrum and can be averaged as above due to isotropy. In deriving $\langle\hat{S}_{prim}\rangle$ we have also used:

$$\int d^2\hat{n}Y_{\ell_1m_1}(\hat{n})Y_{\ell_2m_2}(\hat{n})Y_{\ell_3m_3}(\hat{n}) = I_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} \tag{23}$$

The theoretical primordial angular bispectrum can be written as :

$$B^{ijk}_{\ell_1\ell_2\ell_3} = f_{NL}B^{pqr,prim}_{\ell_1\ell_2\ell_3} \tag{24}$$

where

$$B^{pqr,prim}_{\ell_1\ell_2\ell_3} = 2I_{\ell_1\ell_2\ell_3}\int r^2 dr[\beta^p_{\ell_1}(r)\beta^q_{\ell_2}(r)\alpha^r_{\ell_3}(r)+\beta^r_{\ell_3}(r)\beta^p_{\ell_1}(r)\alpha^q_{\ell_2}(r)+\beta^q_{\ell_2}(r)\beta^r_{\ell_3}(r)\alpha^p_{\ell_1}(r)] \tag{25}$$

Using the above form of the theoretical bispectrum, $\langle\hat{S}_{prim}\rangle$ further simplifies to

$$\langle\hat{S}_{prim}\rangle = \frac{1}{f_{sky}}\sum_{ijkpqr}\sum_{\ell_1\ell_2\ell_3}\tfrac{1}{6}\;B^{pqr,prim}_{\ell_1\ell_2\ell_3}(r)(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}B^{ijk}_{\ell_1\ell_2\ell_3}. \tag{26}$$

Now since $\sum_{\ell_1\ell_2\ell_3} = 6\sum_{\ell_1\leq\ell_2\leq\ell_3}\frac{1}{\Delta_{\ell_1\ell_2\ell_3}}$, where $\Delta_{\ell_1\ell_2\ell_3}$, which is 1 when $\ell_1\neq\ell_2\neq\ell_3$, 6 when $\ell_1=\ell_2=\ell_3$, and 2 otherwise.

$$\langle\hat{S}_{prim}\rangle = \frac{1}{f_{sky}}\sum_{ijkpqr}\sum_{\ell_1\leq\ell_2\leq\ell_3}\frac{1}{\Delta_{\ell_1\ell_2\ell_3}}f_{NL}B^{pqr,prim}_{\ell_1\ell_2\ell_3}(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}B^{ijk,prim}_{\ell_1\ell_2\ell_3} \tag{27}$$

6. Proof of Covariance matrix

In this appendix we prove the equivalence between the optimal estimator given by (Babich & Zaldarriaga 2004) and our fast estimator Eqn. (26)

$$\sum_{ijkpqr}\sum_{\ell_1\ell_2\ell_3}\frac{1}{6}B^{pqr,prim}_{\ell_1\ell_2\ell_3}(C^{-1})^{ip}_{\ell_1}(C^{-1})^{jq}_{\ell_2}(C^{-1})^{kr}_{\ell_3}B^{ijk,prim}_{\ell_1\ell_2\ell_3} = \sum_{\alpha\beta}\sum_{\ell_1\ell_2\ell_3}B^{\alpha,prim}_{\ell_1\ell_2\ell_3}(\mathbf{Cov}^{-1})_{\alpha,\beta}B^{\beta,prim}_{\ell_1\ell_2\ell_3}, \quad (28)$$

This is analogous to the temperature-only case (Komatsu et al. 2005). On the left hand side ijk and pqr run over all the 8 possible ordered combinations $\{TTT, TTE, TET, ETT, TEE, ETE, EET, EEE\}$, while for the estimator on the right hand side α and β run only over the four unordered combinations $\{TTT, TTE, TEE, EEE\}$.

We prove the equivalence between the optimal estimator and the fast estimator for only one combination of ℓ_1, ℓ_2, and ℓ_3, as the proof is same for all the combinations.The covariance matrix **Cov** is obtained in terms of C_ℓ^{TT}, C_ℓ^{EE}, and C_ℓ^{TE} (as in equation 7 in Babich and Zaldarriaga (2004)) by applying Wick's theorem,

$$\mathbf{Cov}_{ijk,pqr} = C^{ir}C^{jq}C^{kp} + C^{iq}C^{jr}C^{kp} + C^{ir}C^{jp}C^{kq} + C^{ip}C^{jr}C^{kq} + C^{iq}C^{jp}C^{kr} + C^{ip}C^{jq}C^{kr}. \quad (29)$$

The covariance matrix above has the form $\mathbf{Cov}_{\alpha\,\beta}$, where α, β, run over $\{TTT, TTE, TEE, EEE\}$, which after simplification gives:

$$\mathbf{Cov}^{-1}_{\alpha\,\beta} = \frac{1}{6}\begin{pmatrix} \hat{C}^{TT}\hat{C}^{TT}\hat{C}^{TT} & 3\hat{C}^{TT}\hat{C}^{TT}\hat{C}^{TE} & 3\hat{C}^{TT}\hat{C}^{TE}\hat{C}^{TE} & \hat{C}^{TE}\hat{C}^{TE}\hat{C}^{TE} \\ 3\hat{C}^{TT}\hat{C}^{TT}\hat{C}^{TE} & 3\hat{C}^{EE}\hat{C}^{TT}\hat{C}^{TT} + 6\hat{C}^{TE}\hat{C}^{TE}\hat{C}^{TT} & 3\hat{C}^{TE}\hat{C}^{TE}\hat{C}^{TE} + 6\hat{C}^{TT}\hat{C}^{EE}\hat{C}^{TE} & 3\hat{C}^{TE}\hat{C}^{TE}\hat{C}^{EE} \\ 3\hat{C}^{TT}\hat{C}^{TE}\hat{C}^{TE} & 3\hat{C}^{TE}\hat{C}^{TE}\hat{C}^{TE} + 6\hat{C}^{TT}\hat{C}^{EE}\hat{C}^{TE} & 6\hat{C}^{EE}\hat{C}^{TE}\hat{C}^{TE} + 3\hat{C}^{TT}\hat{C}^{EE}\hat{C}^{EE} & 3\hat{C}^{TE}\hat{C}^{EE}\hat{C}^{EE} \\ \hat{C}^{TE}\hat{C}^{TE}\hat{C}^{TE} & 3\hat{C}^{TE}\hat{C}^{TE}\hat{C}^{EE} & 3\hat{C}^{TE}\hat{C}^{EE}\hat{C}^{EE} & \hat{C}^{EE}\hat{C}^{EE}\hat{C}^{EE} \end{pmatrix},$$

where $\hat{C}^{XY} \equiv (C^{-1})^{XY}$ are the XY elements of the matrix $\begin{pmatrix} C^{TT} & C^{TE} \\ C^{TE} & C^{EE} \end{pmatrix}^{-1}$. The above matrix is nothing but $\frac{1}{6}(C^{-1})^{ip}(C^{-1})^{jq}(C^{-1})^{kr}$, after the additional permutations with fixed α and β are summed up. For example, for $\alpha = TTE$ the ijk index runs over the set $\{TTE, TET, ETT\}$. This completes the proof.